Math 138 – Practice midterm

Due date: October 15, 2025

Instructions:

- You have 2 hours to complete this exam.
- No external resources are allowed.
- Upload your completed practice midterm to Crowdmark.

Word of advice: While not 100%, we can likely tell if you use AI on this exam. Ultimately the grade on the practice midterm is *completion based*. To cheat on this will likely not result in a higher score on the practice midterm, but will almost certainly result in a lower score on the actual exam. Similarly, the two hour limit is not enforceable, but you should still follow it as it will help you get a better sense of your preparedness for the actual exam.

Question 1. (15 pts)

Spot the error in the following proof, and explain why it is wrong.

Proposition. *Every function $f\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ has a fixed point (i.e., for some x one has $f(x) = x$).*

Proof. We prove this by induction on n. When $n = 1$ the only function $f\colon \{1\} \to \{1\}$ satisfies $f(1) = 1$, so has a fixed point. For the induciton hypothesis, assume the claim is true for any function $g\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, and consider a function

$$f\colon \{1 \ldots, n+1\} \to \{1, \ldots, n+1\}.$$

Case 1: If $f(n+1) = n+1$ we're done.

Case 2: If $f(n+1) \neq n+1$, we can consider the function $g\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ given by $f(i) = g(i)$. By induction hypothesis there is some $x \in \{1, \ldots, n\}$ such that $x = g(x) = f(x)$.

Thus, in either case we see f has a fixed point as desired. $\square$

Question 2. (15 pts)

Let $n \geqslant k \geqslant 0$ be integers. Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Question 3. (20 pts)

Suppose that $x \in \mathbb{Q}$. Use the p-adic valuation to prove that if $x^{\frac{2}{3}}$ belongs to $\mathbb{Q}$ then $x^{\frac{1}{3}}$ belongs to $\mathbb{Q}$.

Question 4. 20 pts

Let A and B be subsets of a set S. Prove that

$$A \triangle B = (A \cup B) \cap (A^c \cup B^c).$$

Question 5. (30 pts)

Prove by induction that if T_n is the number of length $n \geqslant 1$ strings in the symbols $0, 1, 2$ with exactly one 2 then

$$T_n = n \cdot 2^{n-1}$$